# Chapter 4 – Mathematics of Investment

*Objectives:*

At the end of the topic, students are expected to achieve the following:

1. Define and explain the concept of simple interest.
2. Compute simple interests in using different methods.
3. Explain and apply the concept of simple discount.
4. Solve problems involving simple interests and interest rates, with the corresponding principal and length of time.
5. Define and explain the concept of compound interest.
6. Derive formulas for the computation of any required information in a compound interest problem.
7. Solve problems involving compound interest.
8. Identify and distinguish nominal rate and effective rate.
9. Determine the effective rate of a compound interest problem.
10. Apply a specific annuity principle in an appropriate annuity problem.
11. Solve problems involving amortization and sinking fund.

The study on the financial aspect of our day-to-day activities is governed by the different principles under the mathematics of investment. This field of mathematics is responsible for providing details about savings, borrowings, or annuities. These three concepts all involve some amount of money, interests, and interest rates.

In banking and finance, interest and interest rates play a very significant role in boosting or destroying a certain economy. Banks are the main actors in the financial world. Banks provide money as a form of credit, as well as accept money for safekeeping. These two activities of banks always come with a certain amount of money called interest.

For ordinary people, saving a certain amount of money in a savings institution becomes a necessity, especially when one needs to purchase something of higher value. However, if the need is very urgent, where saving money for future use is a bit off the chart, some institutions allow credits with corresponding interests. Loans and installment payments are some forms of these credit activities.

In this chapter, we will be learning things about the interest, interest rate, the borrowed amount, the amount due, and many more terms and activities of the financial market.

## 4.1 SIMPLE INTEREST

When you deposit money in a savings account in a bank, the bank can do whatever it wants with that money. To reward you for that, the bank pays you a certain amount, depending on the amount you deposited. Or when you make a loan from a lending institution, then a fee is charged for that borrowed money. The amount you earn for that deposit or the amount of money paid for that borrowed money is called **interest**. It is usually computed as a *percentage* (called the *interest rate*) of the *principal* (amount invested/deposited or amount borrowed) over a given period of *time* (unless otherwise stated, an annual rate).

### Definition of Terms

**Debtor** or **Maker** – an individual or institution that borrows money for any purpose

**Lender** – an individual or institution which lends money

**Interest** ($I$) – the payment for the use of borrowed money or the amount earned on invested money

**Principal** ($P$) – the amount of money invested or borrowed

**Rate of interest** ($r$) – is the fractional part of the principal that is paid on the loan or investment, which is usually expressed in percent

**Time** ($t$) – the number of years for which the money is borrowed or invested and for which the interest is calculated

**Final Amount** ($F$) **or Maturity Value** ($MV$) – the sum composed of the principal and the interest accumulated over a certain period of time

**Simple Interest** ($I$) – an interest that is charged or paid only on the amount borrowed and not on the past interest

**Simple Interest Formula**

$$I = Prt$$

## Important Points to Remember

1. Interest rates $r$ need to be converted to decimal form first before they are used in any computations that involve it.

2. For converting the time $t$ in days into years, consider the following:
   a. When the ***interest is ordinary***, divide the number of days by $360$.
   b. When ***interest is exact***, divide the number of days by $365$.

3. For converting the time in months into years, always divide the number of months by $12$.

4. Short-term bank loans have terms, between two calendar dates, stated in days. There are two ways to obtain the number of days between these calendar dates:
   a. ***Actual time*** – refers to the exact number of days in a month on a regular calendar.
   b. ***Approximate time*** – refers to the assumption that each month has $30$ days in it.

5. When the interest is not specified in the problem, it is assumed to use the Banker's Rule (ordinary interest with actual time).

6. Four methods to determine the interest between calendar dates:
   a. Ordinary Interest with Actual Time (Banker's Rule)
   b. Ordinary Interest with Approximate Time
   c. Exact Interest with Actual Time
   d. Exact Interest with Approximate Time

7. To make any computations easier and with computational/rounding errors lesser, please do the following:
   a. you must identify first the given information in the problem and represent them into their corresponding letter representations
   b. identify what is asked in the problem
   c. determine the appropriate formula to use
   d. solve the problem by substituting the given information into the formula
   e. units must be placed to all final answers
   f. if the final answer contains more decimals, round it off up to two decimal places only
   g. do not round off any value in between computations, just round off only the final answer
   h. always put your final answer in a box, with the corresponding symbol or unit

## Interests

**Example 1**. You decided to deposit $3{,}500\ baht$ into your savings account that pays $1.75\%$ simple interest. How much interest will it earn after $1\ year$? After $2\ years$? After $3\ years$?

*Given:*

*Required:*

*Formula/s:*

*Solution:*

**Example 2** Danna's mother issued a check in the amount of ฿ $12{,}500$, to settle her loan of ฿ $10{,}500$ that she got two years ago. How much simple interest rate was she charged?

*Given:*

*Formula:*

*Solution:*

*Required:*

**Example 3**. Nine months after borrowing money, Mild pays an interest of ฿ $350$. How much did she borrow if the simple interest rate is $4.5\%$?

*Given:*

*Formula:*

*Solution:*

*Required:*

**Example 4**. Find the ordinary and exact interest on a ฿ $3{,}700$ loan for $180\ days$ at $12\%$ simple interest.

*Given:*

*Formula:*

*Solution:*

*Required:*

**Example 5**. Using the four methods in determining the interest between calendar days, find the interest of ฿ $9{,}700$ at $5\frac{3}{8}\%$ simple interest from $12^{th}$ of July, 2015 to $5^{th}$ of March, 2016.

## Exercise 4.1

Solve the following problems. Don't forget to provide a summary of the given information, the required, the appropriate formula, and a clear solution.

1. Roy borrowed ฿ $8{,}250$ from a lending institution, payable for $120\ days$. What will be the interest and the amount at the end of the term if the simple interest rate is $6\%$?

2. Find the interest on a loan of ฿ $4{,}700$ at $4.5\%$ from June 6, 2017, to November 5, 2017, using the four methods of computing simple interest.

3. Bryan borrowed ฿ $22{,}000$ from AimCorp at $9.2\%$ simple interest for $3\ years$ and $9\ months$. How much did Bryan pay back the corporation at the end of the term?

4. Lawrence borrows ฿ $13{,}000$ from a friend and agrees to pay ฿ 15,500 in $18\ months$. What simple interest rate was he paying?

## Amount or Maturity Value

When someone borrows money with interest, he (the debtor) has to pay what he borrowed (known as the ***principal*** $\boldsymbol{P}$) at the end of the ***term*** $\boldsymbol{t}$, plus the corresponding amount of ***interest*** $\boldsymbol{I}$. The amount that he has to pay is known as the ***final amount*** $\boldsymbol{F}$ or the ***maturity value*** $\boldsymbol{MV}$. Consider the formulas below:

$$\boldsymbol{F = P + I}$$

$$\boldsymbol{F = P + Prt}$$

$$\boldsymbol{F = P(1 + rt)}$$

**Example 6**. Elmo invested his ฿ $50{,}000$ to a corporation that pays $4.5\%$ simple interest for $7\,years$. What is the interest of his investment at the end of the term? How much will he receive at the end of $7\,years$?

**Example 7**. How much should Tim deposit in a bank that pays $2.75\%$ simple interest, to have ฿ $100{,}000$ in $10\,years$ and $3\,months$?

**Example 8**. Jay wants to borrow ฿ $40{,}000$ from a bank that charges a simple interest rate of $3.25\%$. However, he wants to borrow the fund for nine months only. That means he will be able to pay the bank at the end of $9\, months$. How much interest is he going to pay? How much will he have to pay to the bank at the end of the term?

**Example 9**. The repayment on a loan was ฿ $16{,}275$. If the loan was for $15\, months$ and charged at $6.8\%$ simple interest, how much was the principal?

**Example 10**. Ralph wants to have ฿ $100{,}000$ at the end of $10\, years$ and $3\, months$. His choice is to deposit a fixed amount in a corporation that pays $4.25\%$ simple interest. How much should his fixed deposit be?

**Example 11**. Jraz borrowed ฿ $15{,}000$ from her aunt and agreed to pay ฿ $18{,}000$ $18\,months$ after. What is the interest rate involved in this transaction?

**Example 12**. How long will it take for ฿ $55{,}000$ to become ฿ $80{,}000$ if invested in a savings institution that pays $5\frac{6}{7}\%$ simple interest?

**Example 13**. To acquire his dream sports car, Elmo agrees to pay, through equal monthly installments, the amount of the car plus the simple interest for a period of $20\,months$. If the car is worth ฿ $3{,}450{,}000$ and the simple interest is set at $8\frac{5}{7}\%$, how much will his monthly payment be?

## Exercise 4.2

Solve the following problems. Don't forget to provide a summary of the given information, the required, the appropriate formula, and a clear solution.

1. Bryan wants to invest his ฿ $25,000$ into a financial institution that pays $6.5\%$ simple interest. How much will he receive from this investment after $6\frac{1}{2}\ years$?

2. T. Jeryl invested her ฿ $30,000$ at Joe's Savings and Merchandise Corporation. How long will it take for her money to accumulate to ฿ $50,000$ if the simple interest of the corporation is set at $7\frac{6}{7}\%$?

3. T. Jesfer wants to have ฿ $150,000$ in $5\ years$. One of his options is to invest his ฿ $100,000$ money in a savings company. At what simple interest rate this company is paying for this investment?

4. How much should T. Daniel invest today in a time deposit with $3.68\%$ simple interest if he expects to have ฿ $1{,}000{,}000$ for his daughter's university education at the end of $13\ years$?

5. At what simple interest rate will a sum of money double itself after $6\ years$?

6. Josh decided to invest his ฿ $2.5\ million$ to fund a certain committee of his company. How long will it take for this investment to accumulate to ฿ $3.12\ million$ if the bank is paying a simple interest rate of $4.8\%$?

7. A printing company would like to invest a certain amount to a bank, for them to have ฿ $125{,}000$ in $440\ days$, in preparation for the upcoming replacement of their old printing machine. If the bank offers a simple interest rate of $4\,\frac{3}{4}\%$, how much must be invested at the start of the term?

8. What rate was applied to a $6-year$ loan of ฿ $420{,}000$ that accumulates to ฿ $575{,}080$?

9. An amount of ฿ $32{,}500$ is invested at $4.6\%$ simple interest for $5\,years$. Complete the entries of the table below.

| Time | Principal | Rate of Interest | Interest | Maturity Value |
|---|---|---|---|---|
| 0 | ฿ $32{,}500$ | $0.046$ | $0$ | ฿ $32{,}500$ |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |

10. What will be the maturity value of ฿ $18{,}300$ if it is borrowed at $7.5\%$ simple interest for $10\,months$?

## Discount

Another way to calculate interest is by the use of the **discount method**, also referred to as the **bank discount**. This method computes interest based on the maturity value or final amount at the beginning of a specified period of time. The **discount** $\boldsymbol{D}$ in this method is known as the advance payment.

$\boldsymbol{D}$ – discount or payment/interest in advance

$\boldsymbol{F}$ – the amount of the loan or the maturity value

$\boldsymbol{P}$ – proceeds or the amount received after the discount is taken

$\boldsymbol{d}$ – simple discount

$\boldsymbol{t}$ – time or term of discount

Discount Formula:

$$\boldsymbol{D = Fdt}$$

Proceeds Formula:

$$\boldsymbol{P = F - D}$$

**Example 14**. A loan with a ฿ $30{,}000$ total amount is good for $219\ days$. If the simple discount rate is $9\frac{3}{8}\%$, find the proceeds of this loan.

**Example 15**. A borrowed money is discounted by ฿ $1{,}380$. If the simple discount rate is $5\frac{3}{5}\%$, what is the maturity value and the proceeds of this transaction?

**Example 16**. How much loan should you apply for if you need ฿ $35{,}000$ cash that is to be repaid in $8\ months$ at a discount rate of $7\%$?

**Example 17**. Faye borrowed ฿ $8{,}000$, payable for 7 months, from Mary. The simple discount charged in this transaction is $8.35\%$. how much did Faye receive?

## Exercise 4.3

Solve the following discount problems.

1. Tom wants to borrow ฿ $23{,}000$ for the purchase of a new mobile phone. The loan will be payable in $2\ years$ and $5\ months$ from a bank that charges $6.35\%$ simple interest-in-advance. What is the size of the loan that he should apply for?

2. Jerry borrowed ฿ $110{,}000$ from a bank for a term of $4\ years$. How much are the proceeds of the loan if the bank charged a $10.5\%$ simple discount rate?

3. Find the amount of a loan that is payable for $1\ year$ and $8\ months$ at an $8.8\%$ discount rate if the proceeds are ฿ $18{,}500$.

4. How long should a ฿ $30{,}200$ be discounted to have the proceeds of ฿ $21{,}800$, with a $6.5\%$ simple discount rate?

## 4.2 COMPOUND INTEREST

In the present times, several financial institutions, if not all, have been using the method on compound interest. Even the most renowned banks or the most profitable business organizations used the concepts of compound interest.

**Compound interest** is a method of computing interests by the use of every term's ending balance as the basis in computing the next term's beginning value. This means that the interest earned from the previous computation will also earn an interest in the computation. This tells us that as time goes by, the amount continues to grow on a much larger scale, compared to how money grows using the simple interest.

Compound interest is based on a periodic computation, known as the **compounding period $\boldsymbol{m}$**. The following are the values of the compounding periods per year:

1. $m = 1$ if compounding period is annually
2. $m = 2$ if compounding period is semiannually
3. $m = 4$ if compounding period is quarterly
4. $m = 12$ if compounding period is monthly

***Notations:***

| | | |
|---|---|---|
| $\boldsymbol{F}$ | - | final amount or maturity value |
| $\boldsymbol{P}$ | - | principal |
| $\boldsymbol{r}$ | - | annual rate of interest |
| $\boldsymbol{t}$ | - | time or term |
| $\boldsymbol{m}$ | - | number of compounding periods per year |
| $\boldsymbol{i}$ | - | rate per compounding period or $\frac{\boldsymbol{r}}{\boldsymbol{m}}$ |
| $\boldsymbol{n}$ | - | total number of compounding periods or $\boldsymbol{t} \times \boldsymbol{m}$ |

***Computing the compound amount using the simple interest:***

$\boldsymbol{F = P(1 + rt)}$

$\boldsymbol{F = P(1 + i)}$ - end of $1st$ period

$\boldsymbol{F = [P(1 + i)](1 + i) = P(1 + i)^2}$ - end of the $2nd$ period

$\boldsymbol{F = [P(1 + i)^2](1 + i) = P(1 + i)^3}$ - end of the $3rd$ period

$\boldsymbol{F = [P(1 + i)^3](1 + i) = P(1 + i)^4}$ - end of the $4th$ period

.
.
.

$\boxed{\boldsymbol{F = P(1 + i)^n}}$ - end of the $nth$ period

**Example 18**. Suppose you deposit ฿ $5{,}000$ in a bank for $4\ years$. Compare the interests earned if one bank pays $10\%$ simple interest and the other pays $10\%$ compounded annually.

Given:

Given:

| *Simple Interest* | | | |
|---|---|---|---|
| ***t*** | ***P*** | ***I*** | ***F*** |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

| *Compound Interest* | | | |
|---|---|---|---|
| ***t*** | ***P*** | ***I*** | ***F*** |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

**Example 19**. Find the compound amount on a ฿ $53{,}000$ for $10\ years$ at $5.3\%$ compounded quarterly.

### *Derivation on the formulas of $P$, $r$, and $t$:*

$F = P(1+i)^n$

**Principal Formula:**

**Interest Rate Formula:**

**Time/Term Formula:**

**Example 20**. On 8th March 2010, Beth borrowed ฿ $15{,}000$ and promise to pay the principal plus interest at $3\%$ compounded quarterly on 17th November 2014. How much will she pay?

**Example 21**. What amount should be invested to have ฿ $150{,}000$ after $11\,years$ and $6\,months$ if money is worth $9\%$ compounded semiannually?

**Example 22**. A motorcycle was bought on an installment basis with ฿ $3{,}000$ down payment and the balance of ฿ $73{,}000$ in $30\,months$. What is the cash price of the motorcycle if the debt is worth $12.5\%$ compounded monthly?

**Example 23**. What rate compounded quarterly will make ฿ $18{,}000$ become ฿ $25{,}000$ if it is invested for $6\ years$?

**Example 24**. What rate compounded semiannually will double any amount of principal if it is invested in $8\ years$?

**Example 25**. When will ฿ $23{,}000$ earn an interest of ฿ $8{,}000$ if it is invested at $6\frac{3}{7}\%$ compounded quarterly?

**Example 26**. If a principal is invested at a rate of $12.5\%$ compounded semiannually, how long will it take for this principal to double itself?

**Example 27**. In how many years will ฿ $16{,}550$ accumulate to ฿ $24{,}350$ if it is invested at $10\frac{2}{5}\%$ compounded semiannually?

**Example 28**. At what rate compounded monthly should ฿ $48{,}000$ be deposited in a bank to gain the interest of ฿ $15{,}000$ in $5\ years$?

## Exercise 4.4

Solve the following compound interest problems.

1. If ฿ $15{,}000$ is invested at $7\%$ compounded (a) annually, (b) semiannually, (c) quarterly, and (d) monthly, what is the amount after $8\ years$?

2. If ฿ $23{,}000$ amounts to ฿ $37{,}000$ in $4\ years$ and $4\ months$, what is the rate of this investment if the rate is compounded quarterly?

3. Jay wants to have ฿ $150{,}000$, $7\ years$ from now. How much should he invest now to have that amount at his target date if the investment rate is $6\frac{7}{8}\%$ compounded quarterly?

4. Elmo's time deposit account has a value of ฿ $37{,}500$ when he checked it yesterday. He wants to have ฿ $52{,}000$ for his purchase of important equipment. How long will he have to wait, to have the amount he wants, if his deposit is earning $8.2\%$ compounded monthly?

5. Bryan deposited ฿ $7{,}500$ in a bank that pays $2\%$ interest compounded monthly. If no withdrawal is made, how much does he have in his account after $7\ years$ and $5\ month$?

6. How much is the interest earned at the end of $3\ years$ and $7\ months$ from a deposit of ฿ $18{,}000$ if it is invested in a financial institution that pays $7.8\%$ compounded monthly?

7. If money can be invested at $8\frac{3}{4}\%$ compounded quarterly, find the present value of ฿ $90{,}000$ which is due after $9\ years$ and $3\ months$?

8. A ฿ $75{,}000$ capital earned an interest of ฿ $18{,}750$ at the end of $4\,years$. At what rate of interest, compounded semiannually, was the capital invested?

9. When will ฿ $17{,}000$ grow to ฿ $25{,}000$ if it is invested at $6\frac{3}{4}\%$ compounded quarterly?

10. If ฿ $378{,}450$ is the maturity value of an amount invested at $3.45\%$ compounded monthly for $9\,years$ and $6\,months$, how much was the initial deposit? How much is the total interest earned from this deposit?

## Effective Rate

In investment, there are times where we need to choose which of the different offers will give us a higher benefit at the end of the term. What we need is to determine the real return on a savings account or any interest-paying investment. The process of getting these real returns is known as the **effective rate** or **effective annual interest rate**. This reveals the real percentage rate owned in the interest of a loan, a credit card, or any other debt.

What does the effective rate tell us? A bank certificate of deposit, a savings account, or a loan offer may be advertised with its nominal interest rate as well as with its effective annual interest rate. The nominal interest rate does not reflect the effects of compounding interest or even the fees that come with these financial products. Thus, the best way to determine the real return is by the computation of the effective rate.

**Effective Rate Formula:**

$$r_e = \left(1 + \frac{r}{m}\right)^m - 1$$

**Nominal Rate Formula:**

**Example 29**. Which one will give a better investment return: an investment that pays $12\%$ compounded monthly or an investment that pays $12.4\%$ compounded quarterly?

**Example 30**. Determine the effective rate of interest $r_e$ for each of the nominal rates, $r$ compounded $m$ times a year.

| $r$ | $m$ | $r_e$ |
|---|---|---|
| $9\%$ | $4$ | |
| $11\%$ | $2$ | |
| $10\%$ | $3$ | |
| $7\%$ | $12$ | |

## Exercise 4.5

Solve the following effective rate problems.

1. An investor has an opportunity to purchase two different notes: Note $A$ pays $15\%$ compounded monthly, and Note $B$ pays $15.5\%$ compounded quarterly. Which note pays a better investment return?

2. What is the effective rate of interest for money invested at (a) $12\%$ compounded quarterly, and (b) $10\%$ compounded monthly?

3. What is the nominal interest rate compounded monthly for money invested at (a) $7.5\%$ effective rate, and (b) $12\%$ effective rate?

## 4.3 SIMPLE ANNUITIES

Most of the time, when we want something a little bit expensive, the problem is about money. That is, it might be difficult for us to produce the amount immediately. What we usually do is save up so that we will have the exact amount at some future time. However, when we have the money now, the thing we might need last time will not be the thing we might be needing now. This happens especially with technologies.

For example, you might want the latest model of Huawei phones now, which is the Huawei P40 Series. The problem is, you don't have enough money to buy it right away. So, you might think of saving an amount day after day until you'll have the amount needed to purchase the product. But when you have the money (in a year probably), a new model came out, and it's more expensive than the ones you are preparing for last year. The most viable solution for these types of problems is for you to get a phone immediately, though you don't have enough money, through credits. Often, you will pay the balance with equal payments at equal intervals of time. This example is a concept of the annuity.

### Definition of Terms

**Annuity** – a series of periodic payments made at a regular interval of time to pay a loan or create a fund.

**Simple Annuity** – a type of annuity where the interest conversion or compounding period is equal or the same as the payment interval.

**Payment Interval** – the period of time between consecutive payment periods.

**Term of an Annuity** – the time from the beginning of the first payment interval to the end of the last payment interval. This is generally known as the time of an annuity.

**Periodic Payment** – the amount paid during each payment period

**Present Value** of an annuity – is the cash value of all of the future annuity payments. This means the taking of the value of investment now than the value of the same investment in the future.

**Amount** of an annuity – is the total or the final amount of all the periodic payments of the annuity at the end of the term.

### *Types of Annuities:*

1. **Annuity Certain** – is an annuity that is payable for a definite duration. It begins and ends on a definite time or fixed date.

2. **Annuity Uncertain/Contingent** – is an annuity that is payable for an indefinite duration. The beginning of the termination of the periodic payments is dependent on a certain event that cannot be determined accurately.

### *Classifications of Simple Annuities:*

1. **Ordinary Annuity** – is an annuity in which the periodic payment is made at the *end* of each payment interval.

2. **Annuity Due** – is an annuity in which the periodic payment is made at the *beginning* of each payment interval.

3. **Deferred Annuity** – an annuity in which the periodic payment is made not at the beginning or at the end of each payment interval but at *some later time*.

### *Notations:*

| | | |
|---|---|---|
| $\boldsymbol{A}$ | - | present value of an annuity |
| $\boldsymbol{S}$ | - | amount or future value of an annuity |
| $\boldsymbol{R}$ | - | periodic payment |
| $\boldsymbol{r}$ | - | annual rate of interest |
| $\boldsymbol{t}$ | - | time or term |
| $\boldsymbol{m}$ | - | number of compounding periods per year |
| $\boldsymbol{i}$ | - | rate per compounding period or $\frac{r}{m}$ |
| $\boldsymbol{n}$ | - | total number of compounding periods or $\boldsymbol{t} \times \boldsymbol{m}$ |

## Simple Ordinary Annuity

**Present Value Formula:**

$$A = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

**Future Value Formula:**

$$S = R\left[\frac{(1+i)^{n}-1}{i}\right]$$

***Derivation of the formula for $R$, $r$ and $t$:***

**Example 31**. Find the amount and the present value of an ordinary annuity of ฿ $500$ quarterly payment for $7\ years$ and $9\ months$, if money is worth $8\%$ compounded quarterly.

**Example 32**. If you deposit ฿ $500$ at the end of each month on a savings account that pays interest at $4\frac{5}{7}\%$ compounded monthly, how much money do you have after $12\ years$?

**Example 33**. A Yamaha MT-07 big-bike is offered for sale at ฿ $30{,}000$ down payment and then a monthly installment of ฿ $12{,}000$ for the remaining balance for $2\ years$ and $5\ months$. If interest is computed at $6\frac{3}{4}\%$ compounded monthly, what is the cash equivalent of the big-bike?

**Example 34**. Andy obtained a ฿ $150{,}000$ loan to pay his university tuition fee. He has to repay this loan through monthly payments of $20\ months$. How much must his monthly payment be if the interest rate is $7.2\%$ compounded monthly?

**Example 35**. A father invested ฿ $1{,}500{,}000$ in a financial insurance company that pays $4.6\%$ for the monthly allowance of his daughter for the next $15\ years$. How much will the monthly allowance be?

**Example 36**. Claude wants to accumulate ฿ $100{,}000$ by making deposits of ฿ $500$ at the end of every month in a savings account that gives an interest of $2.5\%$ compounded monthly. How many deposits should he make to attain this investment goal?

**Example 37**. Josh wants to save ฿ $120{,}000$ in a period of $5\ years$. If his deposits are made at the end of each month at the amount of ฿ $1{,}500$, how much is the interest rate compounded monthly for this investment?

**Example 38**. T. Julian is interested in buying a condominium that is worth ฿ $2{,}500{,}000$. A down payment of ฿ $500{,}000$ needs to be made first and then a monthly payment of ฿ $23{,}500$ for the balance, with an interest of $4.8\%$ compounded monthly.

(a) How many monthly payments he must make to fully pay the condominium?

(b) How much must his final payment be, if it is made on the same day of the last full payment of ฿ $23{,}500$?

(c) How much must his final payment be, if it is made one month after the last full payment of ฿ $23{,}500$?

## Exercise 4.6

Solve the following simple ordinary annuity problems.

1. Determine the amount and the present value of an ordinary annuity of ฿ $350$ at the end of each quarter, payable for $5\,years$ and $6\,months$, and money is worth $11\%$ compounded quarterly.

2. On Jay's $25th$ birthday, he created a savings account in a bank that pays $6.75\%$ compounded annually. He put up a schedule to deposit ฿ $6{,}000$ on his every birthday up to his $60th$ birthday. How much will be in his account at the end of the term?

3. A home theater is offered for sale at ฿ $85{,}000$ down payment and a ฿ $5{,}300$ payment every three months for the balance, for $5\,years$ and $6\,months$. If interest is computed at $5\%$ compounded quarterly, what is the cash price equivalent of the home theater?

4. A second-hand car is for sale for ฿ $400{,}000$ in cash or on terms of ฿ $80{,}000$ down payment and then ฿ $12{,}000$ monthly payment for the next $24\,months$ with an interest of $5\%$ compounded monthly. If you are the buyer of this car, which purchase plan would you prefer?

5. How much must be paid every end of six months for $20\,years$ to settle an obligation of ฿ $50{,}000$, if money is worth $6\frac{3}{8}\%$ compounded quarterly?

6. Danna deposits ฿ $30{,}000$ at the end of every quarter to save up to ฿ $550{,}000$ for the capital of a new project, $7\,years$. To achieve this goal, at what rate of interest, compounded quarterly, must she invest her money?

7. Jay has already saved ฿ $5,000$ in his account today. Suppose he continues to make ฿ $250$ contributions at the end of each month for the next $14\,years$. For him to achieve his goal of having ฿ $100,000$ at the end of the term, at what rate, compounded monthly, will he put his money in?

8. Janine borrows ฿ $133,830$ at an interest of $4\%$ compounded semiannually. She plans to make an ฿ $18,000$ loan payment at the end of each six months. How many payments must she make to fully pay her loan?

9. How long will it take for a ฿ $90,000$ loan to be paid, if payments of ฿ $2,500$ at the end of each month are made with an interest rate of $6\%$ compounded monthly?

10. A BMW 320D sports car model is up for sale at ฿ $2{,}000{,}000$. Elmo bought the car by paying a down payment of ฿ $300{,}000$ and then monthly payments of ฿ $20{,}000$ for the balance. If the interest rate is $2\frac{3}{5}\%$ compounded monthly:
    (a) How many monthly payments he must make to fully pay for the car?
    (b) How much must his final payment be, if it is made on the same day as the last full payment of ฿ $20{,}000$?
    (c) How much must his final payment be, if it is made one month after the last full payment of ฿ $20{,}000$?

## Simple Annuity Due

**Present Value Formula:**

$$A = R\left[\frac{1-(1+i)^{-(n-1)}}{i}\right]$$

**Future Value Formula:**

$$S = R\left[\frac{(1+i)^{(n+1)}-1}{i}\right]$$

**Example 39**. Jessie wants to set up an educational fund for her daughter by making deposits of ฿ $10{,}000$ at the beginning of every three months for $10\ years$. If money is worth $2.5\%$ compounded quarterly, how much money is in the fund at the end of $10\ years$?

**Example 40**. A phone can be bought by making a cash payment of ฿ $5{,}000$ and six quarterly payments of ฿ $4{,}000$ each, the first payment is due on the date of purchase. Find the cash equivalent of the phone, if money is worth $6\%$ compounded quarterly.

**Example 41**. Joshua plans to travel to Japan, $3\,years$ from now. He will be needing a sum of ฿ $100{,}000$ for this trip, for his trip will last for about three weeks. How much must he put aside in his travel fund every month, starting now, if money is worth $2.25\%$ compounded monthly?

**Example 42**. Claude wants to accumulate ฿ $100{,}000$ by making deposits of ฿ $500$ at the beginning of every month in a savings account that gives an interest of $2.5\%$ compounded monthly. How many deposits should he make to attain this investment goal?

**Example 43**. How long will it take to accumulate ฿ $85{,}000$ if at the beginning of each month, if a deposit of ฿ $2{,}500$ is made with an interest of $9.35\%$ compounded monthly?

**Example 44**. Jelay wants to have ฿ $150{,}000$ in her bank account for her to have a decent amount to renovate her room, her garden, and her gazebo. She is depositing ฿ $1{,}000$ at the beginning of each month with an interest of $5\%$ compounded monthly. How long does she need to have her deposits going?

**Example 45**. Dexter started his savings deposits with ฿ $1{,}500$ at the beginning of each month. $15\ years$ later, his account contained a total of ฿ $425{,}000$. At what interest rate, compounded monthly, did he make his deposits?

## Exercise 4.7

Solve the following simple annuity due problems.

1. Jason made deposits of ฿ $500$ baht, at the beginning of each month, starting on his $20th$ birthday. If the bank charges $3\frac{3}{7}\%$ interest compounded monthly, how much will be in his account during his $50th$ birthday?

2. Revy made rental payments of ฿ $7{,}500$ per month on the condominium unit he is living in. If he lives in that condominium for $2\,years$ already, how much is the present value of his payments if the interest rate is $4.5\%$ compounded monthly?

3. Christoff plans for a tour to Paris. He found out that he needs a total of ฿ $150{,}000$ for these three weeks of a tour that will happen in $2\,years$ and $4\,months$ from now. If he makes deposits at the beginning of each month to a bank that pays $4.75\%$ compounded monthly, how much must his deposits be?

4. A deposit of ฿ $5{,}000$ is made at the beginning of each quarter, with an interest rate of $8.45\%$ compounded quarterly. How long will it take for this series of deposits to accumulate to ฿ $500{,}000$?

5. Nikko withdrew a total of ฿ $325{,}000$ from his account after he made deposits of ฿ $2{,}500$ at the beginning of each month for $9\ years$. At what interest rate, compounded monthly, his deposits were charged?

## Simple Deferred Annuity

**A deferred annuity** is just a simple ordinary annuity but the first payment is delayed at some time. We call the time of the delay as the **deferred period** $\boldsymbol{d}$. During the deferment period, the borrower will not be paying any amount but the borrowed amount will still earn the corresponding interest. We usually call this period the "grace period" or a "time allowance".

**Present Value Formula:**

$$\boldsymbol{A = R\left[\frac{(1+i)^{-d} - (1+i)^{-n-d}}{i}\right]}$$

**Future Value Formula:**

$$\boldsymbol{S = R\left[\frac{(1+i)^{n} - 1}{i}\right]}$$

**Example 46**. Odette bought a car through a sale promotion. She paid ฿ $50{,}000$ down payment and then pay the balance with a monthly installment of ฿ $25{,}000$, where her first payment will be at the end of the $6th$ month, with the interest rate of $4.5\%$ compounded monthly. What is the cash equivalent of the car if it is to be paid within $8\,years$?

**Example 47**. Find the present value of a loan payable by ฿ $2{,}500$ at the end of each month with the interest of $5\frac{7}{8}\%$ compounded monthly for $6\,years$ and is deferred for $1\,year$.

**Example 48**. Find the amount of the quarterly payments to be made at the end of every three months, the first payment is due at the end of $3\ years$ and $3\ months$ and the last at the end of $7\ years$ and $6\ months$, for a loan of ฿ $25{,}500$, if money is worth $5\%$ converted quarterly.

**Example 49**. Ralph is planning to start a new business. To realize it, he needs to apply for a loan of ฿ $550{,}000$. The bank has offered to help him start the business and provide him the amount he needs. If the loan's interest rate is $2\frac{3}{4}\%$ compounded monthly and needs to be repaid in $25\ years$ with the first payment is to be made at the end of $2\ years$, how much will his monthly payment be?

## Exercise 4.8

Solve the following simple deferred annuity problems.

1. Find the present value of a deferred annuity of ฿ $150$ every six months for $8\ years$ if the first payment is made at the start of the $5th\ year$ and money is worth $7.5\%$ compounded semiannually.

2. A fund is raised for ฿ $100{,}000$ by depositing a series of $120$ monthly payments. If money is worth $8\%$ converted monthly, what is the size of the monthly payments when the first payment is in $18\ months$ from now?

3. What amount must be set aside on a boy's $7th$ birthday, which will provide $10$ semiannual payments of ฿ $15{,}000$ for school expenses, if the first payment is to be made on his $10th$ birthday and if money is worth $9\%$ compounded semiannually?

4. Find the cash equivalent of a property that sells for ฿ $50{,}000$ cash and $50$ monthly payments of ฿ $850$, the first is due at the end of $24\,months$ and money is worth $8\%$ converted monthly.

5. A man plans to buy a second-hand car. He is made to choose a ฿ $135{,}000$ cash or pay ฿ $75{,}000$ down payment and $12$ equal annual payments of ฿ $5{,}000$ with an interest of $9\%$ with the first payment due at the end of $2\,years$. How much will he save if he chooses the better alternative?

## 4.4 AMORTIZATION AND SINKING FUND

Most of the time, when we want something a little bit expensive, the problem is about money. That is, it might be difficult for us to produce the amount immediately. What we usually do is save up so that we will have the exact amount at some future time. However, when we have the money now, the thing we might need last time will not be the thing we might be needing now. This happens especially with technologies.

### Amortization Schedule

***Amortization*** refers to the periodic repayment of a debt, principal, and interest through a sequence of equal periodic payments due at the end of equal intervals of time. The equal payments form an annuity of the simple case, whose present value is the original principal of the obligation.

Amortization is similar to the present value of an annuity. It commonly refers to the periodic payment of a debt or loan, which includes the principal and the interest through a sequence of equal periodic payments or installment payments at equal intervals of time.

*An amortization schedule* is a table that shows how much is paid for the interest as well as shows the outstanding principal or remaining liabilities after each payment period.

**Example 50**. Jraz bought a new Huawei P40 Pro phone through a monthly amortization promo. He paid ฿ 5,000 cash as a down payment and agreed to pay the remaining balance of ฿ 29,000 monthly in one year. If the store charged him for a 3.25% interest compounded monthly, what is the size of his monthly payments?

**Example 51**. A loan of ฿ $5{,}000$ is to be amortized by equal payments at the end of every six months for $3\ years$. If the interest is based on a $6\%$ interest compounded semiannually, construct the amortization schedule.

| **Period of Payment** $(n)$ | **Periodic Payment** $(R)$ | **Interest Paid per Period** $(IP)$ | **Repayment to the Principal** $(RP)$ | **Outstanding Balance/Principal** $(OB)$ **or** $(OP)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| **Total** | | | | |

*** Note: $OB_n = OB_{n-1} - RP_n$

$IP_n = i(OB_{n-1})$

$RP_n = R_n - IP_n$

**Example 52**. Construct the first five rows of the amortization schedule for the problem in example 141.

## Amortization – $OP$ without Schedule

The ***outstanding principal*** $\boldsymbol{OP}$ or remaining liability at any date is the present value of all unpaid periodic payments. When the schedule is expensive, the remaining liability at any date can be obtained by the following short cut methods:

- ***Prospective Method*** – this simple method is used when all payments are regular and equal.

$$\boldsymbol{OP = R\left[\frac{1-(1+i)^{-(n-k)}}{i}\right]}$$

- ***Retrospective Method*** – this method makes use of payments already made. So, the outstanding principal is equal to the difference between the accumulated value of the loan and the accumulated value of the past payments.

$$\boldsymbol{OP = A(1+i)^{k} - R\left[\frac{(1+i)^{k}-1}{i}\right]}$$

$OP = outstanding\ principal$ $\quad k = number\ of\ past\ payments$

$R = periodic\ payment$ $\quad n-k = number\ of\ unpaid\ payments$

$i = interest\ rate\ per\ conversion\ period$ $\quad A = original\ loan$

**Example 53**. Toyota offers a certain car model for ฿ $480{,}000$ cash or $25\%$ down payment and the balance is payable in five years by equal monthly installments. If money is worth $7.3\%$ compounded monthly,

a) What is the regular monthly payment for the car?
b) What is the outstanding balance at the end of $3\ years$?
c) What part of the monthly installment goes to the interest on the $45th$ payments date?
d) What part of the $18th$ monthly payment goes to the interest?

## Exercise 4.9

Solve the following amortization problems.

1. A ฿ $20,000$ debt is being repaid at $12\%$ compounded quarterly for two years. Construct the amortization schedule of this transaction.

2. Jay's housing loan of ฿ $680,000$ is to be amortized by payments at the end of each month for $25\ years$. If money is worth $2\frac{1}{2}\%$ compounded monthly, find:
   a) Hs remaining liability just after the $77th$ payment.
   b) His outstanding liability at the end of $10\ years$/
   c) His outstanding principal before the $110th$ payment.

3. A debt will be discharged by paying ฿ $800$ at the end of three months for $10\ years$. If the payments include all interest at $7\%$ payment quarterly,
   a. Find the original principal of a debt.
   b. What part of the $10th$ payment is a repayment of principal?

4. Jay borrowed ฿ $40{,}000$ with interest payable monthly. He agrees to discharge the debt, interest included, with a monthly payment of ฿ $1{,}500$. When will the last full payment be?

5. Bryan borrowed ฿ $30{,}000$ and agrees to discharge the debt by equal payments at the end of each three months for $6\ years$, where these payments include all interests at $5\%$ payable quarterly. How much must his quarterly payment be?

## Sinking Fund

***A sinking fund*** refers to a savings fund created by investing in equal periodic payments. It is designed to ensure the accumulation of a desired amount of money upon a specific date. It is similar to the future value of an ordinary annuity. In general, it also refers to any accounts that are established for accumulating funds to meet a future obligation or debt.

A sinking fund is formed with a definite end in view, such as big expenses in the future. The amount in the fund at any given period is the total of the periodic payments already made and the interest earned.

$$\boldsymbol{S = R\left[\frac{(1+i)^n - 1}{i}\right]}$$

**Example 54**. An amount of ฿ $10{,}000$ will be needed at the end of $2\,years$. If money is invested at $4\%$ compounded semiannually, find the amount to be paid at the end of each six months, and construct a schedule showing the growth of the fund.

| **Payment Number $(n)$** | **Periodic Payment $(R)$** | **Interest in Fund $(IF)$** | **Accumulated Amount $(AA)$** |
|---|---|---|---|
| 1 | | 0 | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| ***Total*** | | | |

Note:

$IF_n = i(AA_{n-1})$

$AA_n = AA_{n-1} + IF_n + R_n$

**Example 55**. How much must a man place in a fund at the end of each month to have ฿ $9{,}000$ at the end of $3\,years$, if money is worth $4.2\%$ compounded monthly? Construct the first seven rows of the sinking fund schedule of this plan.

**Example 56**. To provide a five-year university education for his son, Elmo decided that he must have ฿ $500{,}000$ at the end of $6\,years$. If he invested his money at a monthly sinking fund with a $5\%$ compounded monthly interest, how much must he place in his fund at the end of each month?

## Exercise 4.10

Solve the following sinking fund problems.

1. A sinking fund will be accumulated by investing ฿ $6{,}000$ at the end of each six months at $7\%$ converted semiannually. At the end of $3\,years$ and $6\,months$, how much of the increase in the fund is interest credited then? Construct the sinking fund schedule for this problem.

2. To provide ฿ $3{,}000{,}000$ for a new car at the end of $10\,years$, Josh will deposit equal amounts in a fund at the end of every month. Find the monthly deposit if money is invested at $4\%$ compounded monthly. Make out the first five rows of the sinking fund schedule for this problem.

3. Parents have set up a sinking fund to have ฿ $120{,}000$ in $15\ years$ for their child's debut birthday party. How much should be paid semiannually into an account paying $7\frac{3}{5}\%$ interest compounded semiannually?

4. Starting on her $24th$ birthday, and continuing up to before her $65th$ birthday, Julie deposits ฿ $2{,}000$ every six months into an individual retirement account. How much will be in her account on her $65th$ birthday, if the account earns $6\frac{3}{7}\%$ interest compounded semiannually?

5. Mika wants to make monthly payments of ฿ $500$ into a sinking fund account. If the account pays $5.5\%$ interest converted monthly, how long will it take until his account contains ฿ $400{,}000$?

**The Number of Each Day of the Year** (non–leap years)

| Day of Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec | Day of Month |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 32 | 60 | 91 | 121 | 152 | 182 | 213 | 244 | 274 | 305 | 335 | **1** |
| **2** | 2 | 33 | 61 | 92 | 122 | 153 | 183 | 214 | 245 | 275 | 306 | 336 | **2** |
| **3** | 3 | 34 | 62 | 93 | 123 | 154 | 184 | 215 | 246 | 276 | 307 | 337 | **3** |
| **4** | 4 | 35 | 63 | 94 | 124 | 155 | 185 | 216 | 247 | 277 | 308 | 338 | **4** |
| **5** | 5 | 36 | 64 | 95 | 125 | 156 | 186 | 217 | 248 | 278 | 309 | 339 | **5** |
| **6** | 6 | 37 | 65 | 96 | 126 | 157 | 187 | 218 | 249 | 279 | 310 | 340 | **6** |
| **7** | 7 | 38 | 66 | 97 | 127 | 158 | 188 | 219 | 250 | 280 | 311 | 341 | **7** |
| **8** | 8 | 39 | 67 | 98 | 128 | 159 | 189 | 220 | 251 | 281 | 312 | 342 | **8** |
| **9** | 9 | 40 | 68 | 99 | 129 | 160 | 190 | 221 | 252 | 282 | 313 | 343 | **9** |
| **10** | 10 | 41 | 69 | 100 | 130 | 161 | 191 | 222 | 253 | 283 | 314 | 344 | **10** |
| **11** | 11 | 42 | 70 | 101 | 131 | 162 | 192 | 223 | 254 | 284 | 315 | 345 | **11** |
| **12** | 12 | 43 | 71 | 102 | 132 | 163 | 193 | 224 | 255 | 285 | 316 | 346 | **12** |
| **13** | 13 | 44 | 72 | 103 | 133 | 164 | 194 | 225 | 256 | 286 | 317 | 347 | **13** |
| **14** | 14 | 45 | 73 | 104 | 134 | 165 | 195 | 226 | 257 | 287 | 318 | 348 | **14** |
| **15** | 15 | 46 | 74 | 105 | 135 | 166 | 196 | 227 | 258 | 288 | 319 | 349 | **15** |
| **16** | 16 | 47 | 75 | 106 | 136 | 167 | 197 | 228 | 259 | 289 | 320 | 350 | **16** |
| **17** | 17 | 48 | 76 | 107 | 137 | 168 | 198 | 229 | 260 | 290 | 321 | 351 | **17** |
| **18** | 18 | 49 | 77 | 108 | 138 | 169 | 199 | 230 | 261 | 291 | 322 | 352 | **18** |
| **19** | 19 | 50 | 78 | 109 | 139 | 170 | 200 | 231 | 262 | 292 | 323 | 353 | **19** |
| **20** | 20 | 51 | 79 | 110 | 140 | 171 | 201 | 232 | 263 | 293 | 324 | 354 | **20** |
| **21** | 21 | 52 | 80 | 111 | 141 | 172 | 202 | 233 | 264 | 294 | 325 | 355 | **21** |
| **22** | 22 | 53 | 81 | 112 | 142 | 173 | 203 | 234 | 265 | 295 | 326 | 356 | **22** |
| **23** | 23 | 54 | 82 | 113 | 143 | 174 | 204 | 235 | 266 | 296 | 327 | 357 | **23** |
| **24** | 24 | 55 | 83 | 114 | 144 | 175 | 205 | 236 | 267 | 297 | 328 | 358 | **24** |
| **25** | 25 | 56 | 84 | 115 | 145 | 176 | 206 | 237 | 268 | 298 | 329 | 359 | **25** |
| **26** | 26 | 57 | 85 | 116 | 146 | 177 | 207 | 238 | 269 | 299 | 330 | 360 | **26** |
| **27** | 27 | 58 | 86 | 117 | 147 | 178 | 208 | 239 | 270 | 300 | 331 | 361 | **27** |
| **28** | 28 | 59 | 87 | 118 | 148 | 179 | 209 | 240 | 271 | 301 | 332 | 362 | **28** |
| **29** | 29 | | 88 | 119 | 149 | 180 | 210 | 241 | 272 | 302 | 333 | 363 | **29** |
| **30** | 30 | | 89 | 120 | 150 | 181 | 211 | 242 | 273 | 303 | 334 | 364 | **30** |
| **31** | 31 | | 90 | | 151 | | 212 | 243 | | 304 | | 365 | **31** |

*** Just add 1 unit to each number, from March to December, if the year in the problem is a leap year…